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HILBERT 3-CLASS FIELD TOWERS OF IMAGINARY CUBIC FUNCTION FIELDS

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Abstract. In this paper we study the infiniteness of the Hilbert 3-class field tower of imaginary cubic function fields.

1. Introduction

Let $k = \mathbb{F}_q(T)$ be a rational function field over the finite field $\mathbb{F}_q$, $\infty = (1/T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. For a finite separable extension F of k, write $\mathcal{O}_F$ for the integral closure of $\mathbb{A}$ in F and H_F for the Hilbert class field of F with respect to $\mathcal{O}_F$ (cf. [3]). Let ℓ be a prime number. Let $F_1^{(\ell)}$ be the Hilbert ℓ-class field of $F_0^{(\ell)} = F$, i.e., $F_1^{(\ell)}$ is the maximal ℓ-extension of F inside H_F, and inductively, $F_{n+1}^{(\ell)}$ be the Hilbert ℓ-class field of $F_n^{(\ell)}$ for $n \geq 1$. Then we obtain a sequence of fields

$$F_0^{(\ell)} = F \subset F_1^{(\ell)} \subset \cdots \subset F_n^{(\ell)} \subset \cdots ,$$

which is called the *Hilbert ℓ-class field tower of F*. We say that the Hilbert ℓ-class field tower of F is infinite if $F_n^{(\ell)} \neq F_{n+1}^{(\ell)}$ for each $n \geq 0$. For a multiplicative abelian group A, write $r_\ell(A) = \dim_{\mathbb{F}_\ell}(A/A^\ell)$, which is called the the ℓ-rank of A. Let $\mathcal{C}l_F$ be the ideal class group of $\mathcal{O}_F$ and $\mathcal{O}_F^*$ be the group of units of $\mathcal{O}_F$. In [4], Schoof proved that the Hilbert ℓ-class field tower of F is infinite if $r_\ell(\mathcal{C}l_F) \geq 2 + 2\sqrt{r_\ell(\mathcal{O}_F^*) + 1}$.

Assume that q is odd with $q \equiv 1 \bmod 3$. By an *imaginary cubic function field*, we always mean a finite (geometric) cyclic extension F over k of degree 3 in which ∞ is ramified. In [1], Ahn and Jung studied the infiniteness of the Hilbert 2-class field tower of imaginary quadratic

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function fields. The aim of this paper is to study the infiniteness of the Hilbert 3-class field tower of imaginary cubic function fields.

2. Preliminaries

2.1. Rédei matrix

Assume that q is odd with $q \equiv 1 \bmod 3$. Fix a generator γ of $\mathbb{F}_q^*$. Write $\mathcal{P}$ for the set of all monic irreducible polynomials in $\mathbb{A}$. Any cubic function field F can be written as $F = k(\sqrt[3]{D})$, where $D = aP_1^{r_1} \cdots P_t^{r_t}$ with $a \in \{1, \gamma\}$ and $P_i \in \mathcal{P}$, $r_i \in \{1, 2\}$ for $1 \leq i \leq t$. Then $F = k(\sqrt[3]{D})$ is imaginary if and only if $3 \nmid \deg D$. Let σ be a generator of $G = Gal(F/k)$. Then we have

$$r_3(\mathcal{C}l_F) = \lambda_1(F) + \lambda_2(F), \tag{2.1}$$

where $\lambda_i(F) = \dim_{\mathbb{F}_3} \left(\mathcal{C}l_F^{(1-\sigma)^{i-1}} / \mathcal{C}l_F^{(1-\sigma)^i}\right)$ for $i = 1, 2$. By the Genus theory, $\lambda_1(F) = t - 1$.

Put $\eta = \gamma^{\frac{q-1}{3}}$. Let $R'_F = (e_{ij})_{1 \leq i,j \leq t}$ be a $t \times t$ matrix over $\mathbb{F}_3$, where $e_{ij} \in \mathbb{F}_3$ is defined by $\eta^{e_{ij}} = (\frac{P_i}{P_j})_3$ for $1 \leq i \neq j \leq t$ and the diagonal entries e_{ii} are defined by the relation $\sum_{i=1}^t r_j e_{ij} = 0$ or $d_i + \sum_{i=1}^t r_j e_{ij} = 0$ according as $a = 1$ or $a = \gamma$. Let $d_i \in \mathbb{F}_3$ be defined by $\deg P_i \equiv d_i \bmod 3$ for $1 \leq i \leq t$. Let R_F be the $(t+1) \times t$ matrix over $\mathbb{F}_3$ obtained from R'_F by adjoining $(d_1 \ \cdots \ d_t)$ in the last row. Then we have $\lambda_2(F) = t - \operatorname{rank} R_F$ ([2, Corollary 3.8]). Let ϑ_F be 0 or 1 according as $a = 1$ or $a = \gamma$. Using the relation $\sum_{i=1}^t r_j e_{ij} = 0$ or $d_i + \sum_{i=1}^t r_j e_{ij} = 0$ according as $a = 1$ or $a = \gamma$, it can be shown that $\operatorname{rank} R_F = 1 - \vartheta_F + \operatorname{rank} R'_F$. Therefore, we have

$$\lambda_2(F) = t - 1 + \vartheta_F - \operatorname{rank} R'_F. \tag{2.2}$$

2.2. Martinet's inequality

For a finite separable extension F of k, write $S_\infty(F)$ for the set of all primes of F lying above ∞.

Proposition 2.1. *Let E and K be finite (geometric) separable extensions of k such that E/K is a cyclic extension of degree ℓ, where ℓ is a prime number not dividing q. Let $\gamma_{E/K}$ be the number of prime ideals of $\mathcal{O}_K$ that ramify in E and $\rho_{E/K}$ be the number of places $\mathfrak{p}_\infty$ in $S_\infty(K)$ that ramify or inert in E. If*

$$\gamma_{E/K} \geq |S_\infty(K)| - \rho_{E/K} + 3 + 2\sqrt{\ell |S_\infty(K)| + (1-\ell)\rho_{E/K} + 1},$$

then the Hilbert ℓ-class field tower of E is infinite.

For $D \in \mathbb{A}$, write $\pi(D)$ for the set of all monic irreducible divisors of D.

COROLLARY 2.2. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field over k. If there is a nonconstant monic polynomial D' such that $3|\deg D'$, $\pi(D') \subset \pi(D)$ and $(\frac{D'}{P_1})_3 = (\frac{D'}{P_2})_3 = (\frac{D'}{P_3})_3 = 1$ for $P_1, P_2, P_3 \in \pi(D) \setminus \pi(D')$, then F has infinite Hilbert 3-class field tower.*

Proof. Put $K = k(\sqrt[3]{D'})$. By hypothesis, P_1, P_2, P_3 and ∞ split completely in K. Hence, $E := FK$ is contained in $F_1^{(3)}$. Applying Proposition 2.1 on E/K with $\gamma_{E/K} \geq 9$ and $|S_\infty(K)| = \rho_{E/K} = 3$, we see that E has infinite Hilbert 3-class field tower. Hence F also has infinite Hilbert 3-class field tower. □

COROLLARY 2.3. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field over k. If there are two distinct nonconstant monic polynomials D_1, D_2 such that $3|\deg D_i$, $\pi(D_i) \subset \pi(D)$ for $i = 1, 2$ and $(\frac{D_1}{P_1})_3 = (\frac{D_1}{P_2})_3 = (\frac{D_2}{P_1})_3 = (\frac{D_2}{P_2})_3 = 1$ for some $P_1, P_2 \in \pi(D) \setminus (\pi(D_1) \cup \pi(D_2))$, then F has infinite Hilbert 3-class field tower.*

Proof. Put $K = k(\sqrt[3]{D_1}, \sqrt[3]{D_2})$. By hypothesis, P_1, P_2 and ∞ split completely in K. Hence, $E := FK$ is contained in $F_1^{(3)}$. By applying Proposition 2.1 on E/K with $\gamma_{E/K} \geq 18$ and $|S_\infty(K)| = \rho_{E/K} = 9$, we see that E has infinite Hilbert 3-class field tower. Hence F also has infinite Hilbert 3-class field tower. □

3. Hilbert 3-class field tower of imaginary cubic function field

Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field, where $D = aP_1^{r_1} \cdots P_t^{r_t}$ with $a \in \{1, \gamma\}$, $P_i \in \mathcal{P}$, $e_i \in \{1, 2\}$ for $1 \leq i \leq t$ and $3 \nmid \deg D$. Since $\mathcal{O}_F^* = \mathbb{F}_q^*$ and $r_3(\mathcal{O}_F^*) = 1$, by Schoof's theorem, the Hilbert 3-class field tower of F is infinite if $r_3(\mathcal{C}l_F) = \lambda_1(F) + \lambda_2(F) \geq 5$. By genus theory, we have $\lambda_1(F) = t - 1$. Hence, if $t \geq 6$, then F has infinite Hilbert 3-class field tower.

3.1. Case $t = 4$

In this subsection we will consider the case $t = 4$ in detail.

THEOREM 3.1. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field with $D = aP_1^{r_1}P_2^{r_2}P_3^{r_3}P_4^{r_4}$. Let ϑ_F be 0 or 1 according as $a = 1$ or $a = \gamma$. If $\operatorname{rank} R'_F \leq 1 + \vartheta_F$, then F has infinite Hilbert 3-class field tower.*

Proof. By (2.1) and (2.2), we have $r_3(\mathcal{C}l_F) = 6 + \vartheta_F - \operatorname{rank} R'_F$. Since the Hilbert 3-class field tower of F is infinite if $r_3(\mathcal{C}l_F) \geq 5$, the result follows immediately. □

EXAMPLE 3.2. *Consider $k = \mathbb{F}_7(T)$. Then $\gamma = 3$ is a generator of $\mathbb{F}_7^*$ and $\eta = 2$. Let $P_1 = T, P_2 = T - 1, P_3 = T^2 + T - 1$ and $P_4 = T^2 - T - 1$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $e_{12} = e_{13} = e_{14} = e_{23} = e_{24} = 0, e_{34} = 1$. Let $F = k(\sqrt[3]{D})$ with $D = P_1P_2^2P_3P_4$. Then $\deg D = 7 \not\equiv 0 \bmod 3$, and the matrix R'_F is*

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

whose rank is 1. Hence, F has infinite Hilbert 3-class field tower.

THEOREM 3.3. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field with $D = aP_1^{r_1}P_2^{r_2}P_3^{r_3}P_4^{r_4}$. Then F has infinite Hilbert 3-class field tower if one of the following conditions holds:*

(1) $\deg P_4 \equiv 0 \bmod 3$ *and* $(\frac{P_4}{P_1})_3 = (\frac{P_4}{P_2})_3 = (\frac{P_4}{P_3})_3 = 1$,
(2) $\deg P_3 \equiv \deg P_4 \equiv 0 \bmod 3$ *and* $(\frac{P_3}{P_1})_3 = (\frac{P_3}{P_2})_3 = (\frac{P_4}{P_1})_3 = (\frac{P_4}{P_2})_3 = 1$.

Proof. It follows immediately from Corollary 2.2 and Corollary 2.3. □

EXAMPLE 3.4. *Let $k = \mathbb{F}_7(T)$. Let $P_1 = T, P_2 = T-1, P_3 = T^2-T-1$ and $P_4 = T^3 + T - 1$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $(\frac{P_4}{P_1})_3 = (\frac{P_4}{P_2})_3 = (\frac{P_4}{P_3})_3 = 1$. Let $D = \gamma P_1P_2P_3P_4$. Hence, the Hilbert 3-class field tower of $F = k(\sqrt[3]{D})$ by Theorem 3.3. But, the matrix R'_F is*

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

whose rank is 3. So Theorem 3.1 can't guarantee the infiniteness of Hilbert 3-class field tower of F.

3.2. Case $t = 5$

In this subsection we will consider the case $t = 5$ in detail.

THEOREM 3.5. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field with $D = aP_1^{r_1}P_2^{r_2}P_3^{r_3}P_4^{r_4}P_5^{r_5}$. Let $R'_F = (e_{ij})_{1\le i,j\le 5}$ be the 5×5 matrix over $\mathbb{F}_3$ given in §2.1. If $a = 1$ with* $\operatorname{rank} R'_F \le 3$ *or $a = \gamma$ with* $\operatorname{rank} R'_F \le 4$, *then F has infinite Hilbert 3-class field tower.*

Proof. By (2.1) and (2.2), we have $r_3(\mathcal{C}l_F) = 8 - \operatorname{rank} R'_F$ or $r_3(\mathcal{C}l_F) = 9 - \operatorname{rank} R'_F$ according as $a = 1$ or $a = \gamma$. Since the Hilbert 3-class field tower of F is infinite if $r_3(\mathcal{C}l_F) \ge 5$, the result follows immediately. $\square$

EXAMPLE 3.6. *Let $k = \mathbb{F}_7(T)$. Let $P_1 = T, P_2 = T-1, P_3 = T^2-T-1, P_4 = T^3+T-1$ and $P_5 = T^3+T-1$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $e_{12} = e_{13} = e_{14} = e_{15} = e_{23} = e_{24} = e_{25} = e_{45} = 0, e_{34} = e_{35} = 1$. Let $F = k(\sqrt[3]{D})$ with $D = P_1P_2^2P_3P_4P_5$. Then* $\deg D = 10 \not\equiv 0 \bmod 3$, *and the matrix R'_F is*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix}$$

whose rank is 2. *Hence, F has infinite Hilbert* 3*-class field tower.*

THEOREM 3.7. *Assume that q is odd with $q \equiv 1 \bmod 3$. Let $F = k(\sqrt[3]{D})$ be an imaginary cubic function field with $D = aP_1^{r_1}P_2^{r_2}P_3^{r_3}P_4^{r_4}P_5^{r_5}$. Then F has infinite Hilbert* 3*-class field tower if one of the following conditions holds:*

(1) $\deg P_5 \equiv 0 \bmod 3$ *and* $(\frac{P_5}{P_1})_3 = (\frac{P_5}{P_2})_3 = (\frac{P_5}{P_3})_3 = 1$,

(2) $\deg P_4 \equiv \deg P_5 \equiv 0 \bmod 3$ *and* $(\frac{P_4}{P_1})_3 = (\frac{P_4}{P_2})_3 = (\frac{P_5}{P_1})_3 = (\frac{P_5}{P_2})_3 = 1$,

(3) $\deg P_3 \equiv \deg P_4 \equiv \deg P_5 \equiv 0 \bmod 3$ *and the rank of* $\left(\begin{smallmatrix} e_{13} & e_{14} & e_{15} \\ e_{23} & e_{24} & e_{25} \end{smallmatrix}\right)$ *is* ≤ 1.

Proof. (1) and (2) follow immediately from Corollary 2.2 and Corollary 2.3, respectively. For (3), by hypothesis, we can choose $x, y, z, w \in \mathbb{F}_3$ such that $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \ne \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \ne \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} e_{13} & e_{14} \\ e_{23} & e_{24} \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} e_{13} & e_{15} \\ e_{23} & e_{25} \end{smallmatrix}\right)\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Note that $e_{ij} = e_{ji}$ for $i \ne j$. We have

$$\left(\frac{P_3^xP_4^y}{P_1}\right)_3 = \eta^{xe_{31}+ye_{41}} = 1, \quad \left(\frac{P_3^xP_4^y}{P_2}\right)_3 = \eta^{xe_{32}+ye_{42}} = 1,$$
$$\left(\frac{P_3^zP_5^w}{P_1}\right)_3 = \eta^{ze_{31}+we_{51}} = 1, \quad \left(\frac{P_3^zP_5^w}{P_2}\right)_3 = \eta^{ze_{32}+we_{52}} = 1.$$

Since $D_1 = P_3^x P_4^y$ and $D_2 = P_3^z P_5^w$ are nonconstant monic polynomials whose degree are divisible by 3, by Corollary 2.3, F has infinite Hilbert 3-class field tower. $\square$

EXAMPLE 3.8. *Let $k = \mathbb{F}_7(T)$. Let $P_1 = T, P_2 = T-1, P_3 = T^3+T-1, P_4 = T^3-3T+1$ and $P_5 = T^3-T-2$, which are all monic irreducible polynomials in $\mathbb{A} = \mathbb{F}_7[T]$. We have $e_{13} = e_{14} = e_{23} = e_{24} = 0$ and $e_{15} = e_{25} = 2$. Let $D = P_1P_2P_3P_4P_5$. By Theorem 3.7, the Hilbert 3-class field tower of $F = k(\sqrt[3]{D})$ is infinite.*

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